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Stability of phase locking in a ring of unidirectionally coupled oscillators

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Abstract

We discuss the dynamic behaviour of a finite group of phase oscillators unidirectionally coupled in a ring. The dynamics are based on the Kuramoto model. In the case of identical oscillators, all phase locking solutions and their stability properties are obtained. For nonidentical oscillators it is proven that there exist phase locking solutions for sufficiently strong coupling. An algorithm to obtain all phase locking solutions is proposed. These solutions can be classified into classes, each with its own stability properties. The stability properties are obtained by means of a novel extension of Gershgorin's theorem. One class of stable solutions has the property that all phase differences between neighbouring cells are contained in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Contrary to intuition, a second class of stable solutions is established with exactly one of the phase differences contained in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. The stability results are extended from sinusoidal interconnections to a class of odd functions. To conclude, a connection with the field of active antenna arrays is made, generalizing some results earlier obtained in this field.

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1. Introduction

Oscillating systems coupled into a ring formation serve as a model for a wide array of applications: gaits of n-legged animals (Golubitsky *et al* 1998), twining of plants (Lubkin 1994), rings of semiconductor lasers (Silber *et al* 1993) and circular antenna arrays (Davies 1983, Dussopt and Laheurte 1999).

Several mathematical models have been proposed and studied in the literature. In Ermentrout (1985) and Laing (1998), each oscillator is modelled as a nonlinear system with an attracting limit cycle and the coupling is bidirectional and of the nearest-neighbour type.

In Ermentrout (1985) the existence and necessary conditions for the stability of phase locking behaviour of the network is established. In Laing (1998) a ring of identical oscillators is considered and a Hopf bifurcation of the network is investigated to obtain different types of stable oscillation. In Golubitsky and Stewart (1988) more general networks of identical oscillators are considered: the symmetries of the network are exploited in order to obtain different types of phase locking behaviour. In Woodward (1990), initial conditions and distributions on the natural frequencies of the oscillators are determined that correspond to a given type of rational frequency ratio.

The model in this paper is based on the model proposed by Kuramoto (1984), although with a different connection structure. We consider a ring structure of N oscillators with *unidirectional* coupling: the ith oscillator is influenced by the $(i + 1)$th oscillator for $i = 1, \dots, N - 1$, and the Nth oscillator is influenced by the first. The oscillators need not be identical.

Rings of unidirectionally coupled oscillators are typically encountered in the modelling of animal locomotion (Luo *et al* 2004, Collins and Stewart 1993). These structures are candidates for rhythmic pattern-generating networks of mutually coupled neurons in the central nervous system. They are called *central pattern generators* (CPG) and control the motion of the limbs of an n-legged animal. In a simplified model, each neuron in the network corresponds to one leg of the animal. The present paper obtains the full picture of this type of networks when modelled by the Kuramoto equations, determining *all* phase locking solutions and their stability properties. Moreover these results are extended to a more general coupling, not just sinusoidal coupling. Note also that we allow for oscillators with distinct natural frequencies of oscillation. Different configurations of natural frequencies lead to different sets of phase locking solutions. In this way it is possible to influence the dynamic behaviour of the system by adjusting the natural frequencies.

The Kuramoto equations are also compatible with models of antenna arrays. Each antenna is steered by a voltage controlled oscillator (VCO) and these oscillators are mutually coupled according to some desired topology. When all VCOs oscillate at the same frequency, their phase differences determine the radiation pattern of the entire array. The present paper determines the stable solutions for the phase differences, when a configuration of the natural frequencies of the VCOs is given for the unidirectional ring topology. A second application in antenna arrays is using the unidirectional ring structure to construct circular polarization of an antenna array (Dussopt and Laheurte 1999). This is explained in section 7.

In a different context of cyclic pursuit problems, patterns remarkably similar to the phase locking solutions obtained in the present paper emerge. In the case of N mobile robots moving in two-dimensional space, where the ith robot follows the $(i + 1)$th and the Nth follows the first, it is shown by Marshall *et al* (2003, 2004) that the robots end up moving in a circle at the same velocity. There exist several qualitatively different stable formations of this kind, mutually distinguishable by the relative positions of the robots on the circle.

2. System dynamics

Each oscillator is a dynamical system with a unique, stable, isolated limit cycle when not coupled with other oscillators, i.e., each oscillator exhibits a periodic behaviour when uncoupled. This behaviour can be captured by a model where the state of the oscillator is one scalar variable θ, called *the phase* of the oscillator (Kuramoto 1984). The phase of the ith oscillator, when uncoupled, evolves in time according to the differential equation

$$\dot{\theta}_i = \omega_i, \quad \theta_i \in S^1, \quad \omega_i \in \mathbb{R},$$

where the *natural frequency* ω_i of the ith oscillator is the angular velocity of the periodic motion. The phase of an oscillator can be visualized as a point moving around on the unit circle.

The system equations of N ($N \in \mathbb{N}$) oscillators unidirectionally coupled in a ring are

$$\dot{\theta}_i = \omega_i + K \sin(\theta_{i+1} - \theta_i), \qquad i \in \underline{N}, \tag{1}$$

with $\underline{N} \triangleq \{1, \ldots, N\}$ and $\theta_{N+1} \equiv \theta_1$. The parameter $K > 0$ is called the *coupling strength*. For the moment, the interaction is implemented by a sine function. See section 6 for an extension.

3. Existence of phase locking solutions: identical oscillators

All oscillators have the same natural frequency ω. After substitution $\theta_i \to \theta_i + \omega t$, the system equations are

$$\dot{\theta}_i = K \sin(\theta_{i+1} - \theta_i), \qquad i \in \underline{N}. \tag{2}$$

Define the phase differences $\phi_i \in S^1 : \phi_i \triangleq (\theta_i - \theta_{i-1}) \bmod 2\pi, i = 2, \ldots, N$ and $\phi_1 \triangleq (\theta_1 - \theta_N) \bmod 2\pi$. A network is called *phase locked* when the phase difference between each pair of oscillators is constant in time.

Theorem 1. *Consider all couples* $(\alpha, m), \alpha \in S^1, m \in \underline{N}$*, satisfying*

$$m\alpha + (N - m)(\pi - \alpha) = 2\pi k, \qquad k \in T,$$

with $T \triangleq \{0, \ldots, N - 1\}$*. Assign to each such couple* (α, m) *the vector*

$$(\underbrace{\alpha, \ldots, \alpha}_{m}, \underbrace{\pi - \alpha, \ldots, \pi - \alpha}_{N-m}). \tag{3}$$

Every vector $\phi = (\phi_1, \ldots, \phi_N)$ *that is a permutation of such a vector (3), corresponds to a phase locking solution of (2).*

Proof. The phase differences $\phi_1, \ldots, \phi_N$ are solutions of the system equations

$$\dot{\phi}_i = K(\sin\phi_{i+1} - \sin\phi_i), \qquad i \in \underline{N}, \tag{4}$$

with $\phi_{N+1} \equiv \phi_1$. The phase locking solutions of (2) are the equilibrium points of (4), hence each phase locking solution satisfies the set of equations

$$(\phi_i - \phi_{i+1})(\phi_i - \pi + \phi_{i+1}) = 0, \qquad \forall i \in \underline{N}. \tag{5}$$

Every combination of either the first factor or the second factor equal to zero in each of these equations corresponds to a phase locking solution. Suppose that one of the phase differences assumes the value $\alpha \in S^1$. From (5) it follows that the other phase differences assume the values α or $\pi - \alpha$. Assume m phase differences equal to α are present. In a phase locking solution the phase differences ϕ_i have to add up to an integer multiple of 2π:

$$\sum_{j=1}^{N} \phi_j = 2\pi k, \qquad k \in T, \tag{6}$$

with $T = \{0, \ldots, N - 1\}$. It follows that

$$m\alpha + (N - m)(\pi - \alpha) = 2\pi k, \qquad k \in T. \qquad \square$$

A solution (3) with $\alpha = 0$ and $m \neq N$ is called *an elementary solution*. The solution $\phi = (0, \ldots, 0)$ is called the *synchronized solution*. The solutions

$$\phi = \left(\frac{2\pi k}{N}, \ldots, \frac{2\pi k}{N}\right), \qquad k \in \{1, \ldots, N - 1\}, \tag{7}$$

are the so-called *travelling wave solutions*.

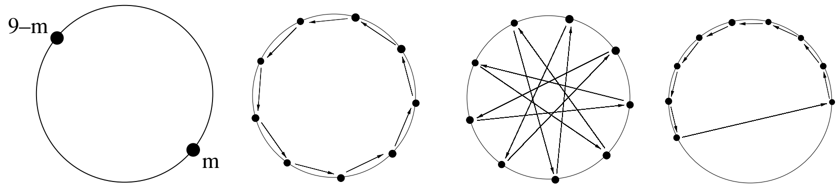

Figure 1. Different types of phase locking solutions for a ring of nine identical oscillators: (*a*) elementary solution, (*b*) travelling wave solutions and (*c*) other type of solution.

Because of (2), constant phase differences for all i imply the constancy of the phase velocities, $\dot{\theta}_i = \Omega_i$. But on the other hand, $\theta_{i+1} - \theta_i = a_i = \text{const}$ implies also $\dot{\theta}_i = \dot{\theta}_{i+1}$, and therefore $\dot{\theta}_i = \Omega, \forall i \in \underline{N}$. The constant Ω is called *the group velocity* of the phase locking solution.

From (2), the group velocity at which a phase locked travelling wave group moves is $\Omega = \sin((2\pi k)/N)K$. Stronger coupled oscillators results in a faster motion, in case of a travelling wave. If the oscillators synchronize, the group velocity Ω is zero. This behaviour is completely different from the case in which the oscillators are bidirectionally coupled, where Ω is independent of K for all phase locking solutions. Note that in the original coordinates, i.e., before the substitution $\theta_i \to \theta_i + \omega t$ at the beginning of this section, the group velocity is $\Omega + \omega$.

Figure 1 shows different types of possible phase locking solutions in a ring of 9 identical oscillators. Each figure is a snapshot taken of the population moving along the circle at the group velocity Ω. The figure on the left-hand side shows all elementary solutions ($m = 1, \ldots, 8$). In this figure it is indicated how many oscillators are represented by each dot on the circle. The middle figure shows two travelling wave solutions. Each dot corresponds to one oscillator. The arrows represent the interconnections between the oscillators. Figure 1(c) is an example of a solution not belonging to the previous two classes.

4. Existence of phase locking solutions: nonidentical oscillators

In this section the dynamics described by (1) is investigated. It turns out that in this case there exist phase locking solutions as well. Assume that $\dot{\theta}_i = \Omega, \forall i \in \underline{N}$. The phase differences have to satisfy

$$\omega_i + K \sin\phi_{i+1} = \Omega, \qquad i \in \underline{N}, \tag{8}$$

or equivalently,

$$\phi_{i+1} = g_i\left(\frac{\Omega - \omega_i}{K}\right), \qquad i \in \underline{N}, \tag{9}$$

where the function g_i can be any of the functions f_0 and f_1, defined as

$$\begin{aligned} f_0 &: [-1, 1] \to (-\pi/2, \pi/2) : t \mapsto \arcsin(t), \\ f_1 &: [-1, 1] \to [\pi/2, 3\pi/2] : t \mapsto \pi - \arcsin(t). \end{aligned} \tag{10}$$

Since the phase differences ϕ_i have to add up to an integer multiple of 2π in a phase locking solution, it holds that

$$g_1\left(\frac{\Omega - \omega_1}{K}\right) + g_2\left(\frac{\Omega - \omega_2}{K}\right) + \cdots + g_N\left(\frac{\Omega - \omega_N}{K}\right) = 2\pi k, \qquad k \in \mathbb{Z} \tag{11}$$

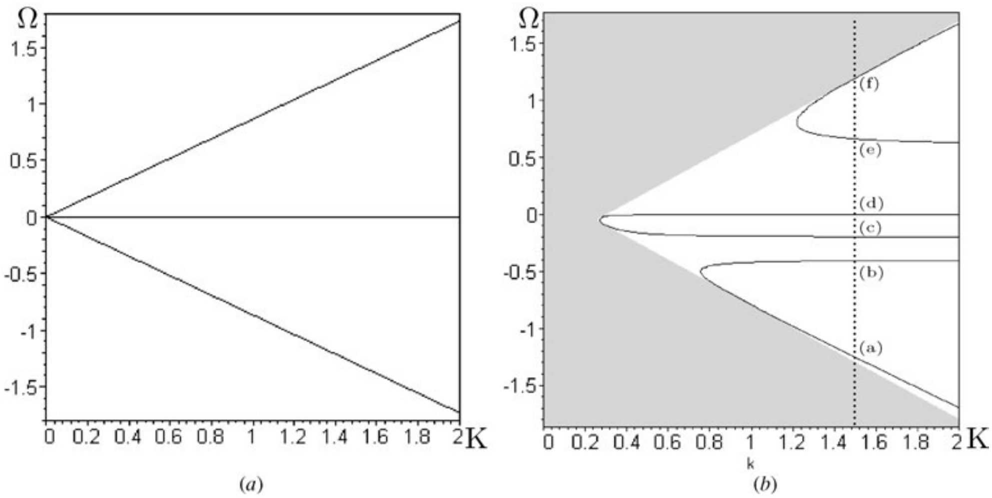

Figure 2. The group velocity $\Omega(K)$ of 2 ring configurations of three oscillators. (The symbols (a)–(f) in figure (*b*) refer to figure 5). (*a*) The group velocity $\Omega(K)$ of all phase locking solutions of the ring of three identical oscillators with $\omega = 0$. (*b*) The group velocity $\Omega(K)$ of all phase locking solutions of the ring of three oscillators with $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = -0.3$.

with every g_i, $i \in \underline{N}$, replaced by f_0 or f_1, leading to 2^N equations for a fixed value of k. Expression (11) is called *the consistency condition on the group velocity*. Each phase locking solution $(\phi_1, \ldots, \phi_N)$ satisfies exactly one of these equations, since the images of f_0 and f_1 are disjunct. Hence, each equation *separately* yields a number of solutions of the group velocity Ω. However, how many solutions correspond to each equation is not known. It is also possible that an equation does not possess any solution at all.

Computing all phase locking solutions is done by investigating the consistency condition. Let the coupling strength assume some value K_1 and let $k = 0$. Then (11) is a set of 2^N equations with Ω the remaining unknown. For each equation the corresponding solutions Ω are determined separately. The solutions Ω computed in this way are the group velocities of those phase locking solutions with phase differences adding up to zero (since $k = 0$) and corresponding with a coupling strength K_1. Once a value of Ω is obtained from (11), the values of the corresponding phase differences can be determined via (9).

This procedure is then repeated for all $k \in \mathbb{Z}$, resulting in all solutions of Ω corresponding to the coupling strength K_1. Performing the above procedure for every K results in a diagram as shown in figure 2. Note that the set of k values for which the computation has to be done can be reduced to a subset of $\mathbb{Z}$. Each phase difference ϕ_i assumes a value in $(-\pi/2, 3\pi/2]$, according to (10). The left-hand side of (11) then assumes a value in $(-\pi N/2, 3\pi N/2]$. Hence, equations of (11) with k not belonging to $\{l \in \mathbb{Z} : (-\pi N)/2 < 2\pi l \leqslant (3\pi N)/2\}$ do not yield phase locking solutions.

In figure 2, $\Omega(K)$ is plotted for two ring configurations. On the left-hand side the group velocity of a ring with three identical oscillators, with $\omega = 0$, is displayed. The group velocity is calculated using (11) and the resulting branches are $\Omega(K) = 0$, $\Omega(K) = K\sin(\pi/3)$ and $\Omega(K) = -K\sin(\pi/3)$. The six corresponding solutions are displayed in figures 3 and 4. The phase differences ϕ_i of each of these solutions are independent of the coupling strength.

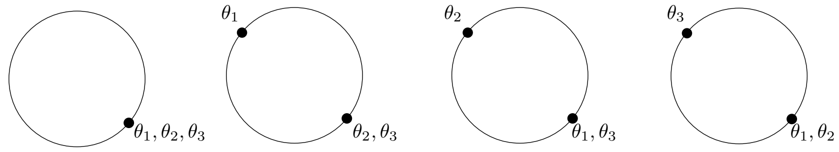

Figure 3. All phase locking solutions corresponding to the central branch (with group velocity zero) of figure 2(*a*).

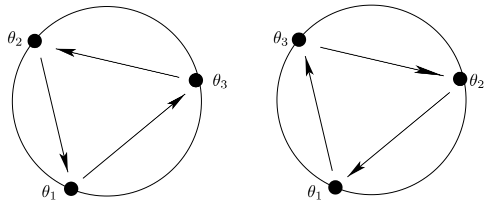

Figure 4. Left: the phase locking solution corresponding to the branch $\Omega(K) = -K\,\sin(\pi/3)$ of figure 2(*a*). Right: the phase locking solution corresponding to the branch $\Omega(K) = K\,\sin(\pi/3)$.

In figure 2(*b*) the case of a ring of three oscillators with $\omega_1 = 0.1, \omega_2 = 0.2$ and $\omega_3 = -0.3$ is shown. The shaded area in this figure is the set of points (K, Ω) that do not satisfy $K \geqslant |\Omega - \omega_i|, \forall i$. For these points it holds that

$$K < K_\Omega \Rightarrow \exists i : |\sin\phi_{i+1}| = \frac{|\Omega - \omega_i|}{K} > 1,$$

hence the shaded area is the area where no phase locking solutions are possible. For small K no phase locking solutions exist. When the coupling strength exceeds some threshold value K_T, phase locking solutions arise. For sufficiently large K, six solutions $\Omega(K)$ exist. Computation of the corresponding phase locking solutions via (9) reveals that there corresponds only one phase locking solution to each solution $\Omega(K)$. Three of the solutions $\Omega(K)$, corresponding to branches (*a*), (*d*) and (*f*), converge in the limit $K \to \infty$ to the solutions of figure 2(*a*). The remaining three branches are solutions of the following three equations of (11):

$$\begin{aligned}
&\arcsin((\Omega - \omega_1)/K) - \arcsin((\Omega - \omega_2)/K) - \arcsin((\Omega - \omega_3)/K) = 0,\\
&-\arcsin((\Omega - \omega_1)/K) + \arcsin((\Omega - \omega_2)/K) - \arcsin((\Omega - \omega_3)/K) = 0,\\
&-\arcsin((\Omega - \omega_1)/K) - \arcsin((\Omega - \omega_2)/K) + \arcsin((\Omega - \omega_3)/K) = 0.
\end{aligned} \tag{12}$$

The limit value of these branches for $K \to \infty$ is analytically determined as follows. Assume that the solution Ω is bounded. Then for $K \to \infty$, the arcsine function can be approximated by its argument and (12) changes into

$$\begin{aligned}
&(\Omega - \omega_1)/K - (\Omega - \omega_2)/K - (\Omega - \omega_3)/K = 0,\\
&-(\Omega - \omega_1)/K + (\Omega - \omega_2)/K - (\Omega - \omega_3)/K = 0,\\
&-(\Omega - \omega_1)/K - (\Omega - \omega_2)/K + (\Omega - \omega_3)/K = 0,
\end{aligned}$$

resulting in three limit values of Ω: $-\omega_1 + \omega_2 + \omega_3$, $-\omega_1 - \omega_2 + \omega_3$ and $\omega_1 + \omega_2 - \omega_3$, which can be observed in figure 2.

For $K = 1.5$ the six phase locking solutions corresponding to figure 2(*b*) are depicted in figure 5. The solutions are ordered by increasing group velocity Ω.

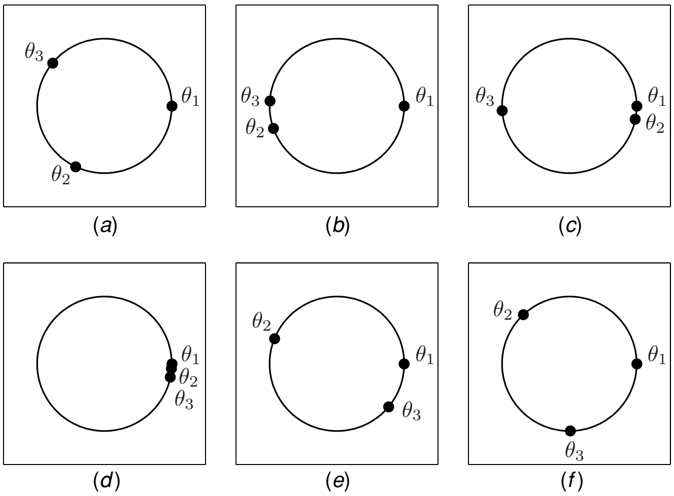

Figure 5. All phase locking solutions of the ring configuration of figure 2(*b*) with coupling strength $K = 1.5$, ordered by increasing group velocity Ω.

5. Stability properties

5.1. Preliminary results

Theorem 2 (Gershgorin's theorem)**.** *For a matrix* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$*, define*

$$R_i \triangleq \sum_{j=1, j\neq i}^{n} |a_{ij}|.$$

Each eigenvalue of A is contained in at least one of the discs D_i, defined by

$$D_i \triangleq \{z : |z - a_{ii}| \leqslant R_i\}.$$

It is important to note that all Gershgorin discs do not necessarily contain at least one eigenvalue. The following theorem gives some more detailed information regarding the eigenvalues of A (Meyer 2000).

Theorem 3. *If $\left(\cup_{j=1}^{m} D_j\right) \cap \left(\cup_{j=m+1}^{n} D_j\right) = \emptyset$, then $\cup_{j=1}^{m} D_j$ contains exactly m eigenvalues of A, with each eigenvalue being counted according to its algebraic multiplicity. The remaining eigenvalues are in $\cup_{j=m+1}^{n} D_j$.*

In order to be able to prove the stability properties of the phase locking solutions, this theorem has been extended by us as follows.

Theorem 4. *If $\left(\cup_{j=1}^{m} D_j\right) \cap \left(\cup_{j=m+1}^{n} D_j\right) = \{p\}$, $p \in \mathbb{C}$, then $\cup_{j=1}^{m} D_j$ contains at least m eigenvalues of A, with each eigenvalue being counted according to its algebraic multiplicity. The region defined by $\cup_{j=m+1}^{n} D_j$ contains at least $n - m$ eigenvalues.*

Proof. Define $A_D = \mathrm{diag}\ (a_{11}, a_{22}, \ldots, a_{nn})$ and let $A = A_D + B$. For $t \in [0, 1]$ define $A_t = A_D + tB$. Then $A_0 = A_D$ and $A_1 = A$. Call the eigenvalues of A_t, λ_t. Because of theorem 3, for $t \in [0, 1)$, $\cup_{j=1}^{m} D_{j,t}$ contains m eigenvalues and $\cup_{j=m+1}^{n} D_{j,t}$ contains the remaining eigenvalues.

Define the discs $D'_{i,t}$ with centres a_{ii} and radii $r_{i,t}$ as

$$D'_{i,t} \in \{D_{j,t}\} \qquad \text{and} \qquad p \in D'_{i,1}.$$

Suppose that for $t = 1, \cup_{j=1}^{m} D_{j,t}$ contains $k < m$ eigenvalues. This implies that

$$\exists j, \exists \lambda_t : \begin{cases} |\lambda_t - a_{jj}| \leqslant r_{i,t} & \text{for} \quad t \in [0, 1), \\ |\lambda_t - a_{jj}| > r_{i,t} & \text{for} \quad t = 1, \end{cases}$$

or equivalently, that there exists a continuous function $g(t) \triangleq |\lambda_t - a_{jj}| - r_{i,t}$:

$$\begin{cases} g(t) \leqslant 0 & \text{for} \quad t \in [0, 1), \\ g(t) > 0 & \text{for} \quad t = 1. \end{cases}$$

This is not possible, so the assumption was incorrect. Similarly it can be proven that $\cup_{j=m+1}^{n} D_j$ contains at least $n - m$ eigenvalues.

Suppose $\cup_{j=1}^{m} D_j$ contains $k > m$ eigenvalues. It then follows that at least $k - m$ eigenvalues are located in p. □

5.2. Linearization

Every phase locking solution of (1) can be transformed into a curve of equilibrium points by applying the appropriate change of coordinates: if the group velocity of the phase locking solution under consideration is Ω, then the new coordinates $\tilde{\theta}$ are defined by $\tilde{\theta}_i \triangleq \theta_i - \Omega t$.

Since only phase differences govern the dynamics, solutions are only unique up to a uniform phase shift. As a consequence, the linearization around each equilibrium point will contain at least one zero-eigenvalue. This zero eigenvalue has no influence on the stability properties.

The linearization about a phase locking solution is the matrix $J \in \mathbb{R}^{N \times N}$, with the phase differences ϕ_i assuming values corresponding to that phase locking solution:

$$J = K \begin{bmatrix} -\cos\phi_1 & \cos\phi_1 & 0 & \cdots & 0 \\ 0 & -\cos\phi_2 & \cos\phi_2 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & -\cos\phi_{N-1} & \cos\phi_{N-1} \\ \cos\phi_N & 0 & \cdots & 0 & -\cos\phi_N \end{bmatrix}. \tag{13}$$

The characteristic polynomial of the linearization is

$$\prod_{j=1}^{N} (\lambda + \cos\phi_j) - \prod_{j=1}^{N} \cos\phi_j, \tag{14}$$

which clearly is invariant under permutations of the phase differences.

Calculation of the Gershgorin discs reveals that each disc lies in a closed half plane and contains the origin. The centres of these discs are given by the diagonal elements of J.

Phase locking solutions with phase differences equal to $\pi/2$ or $3\pi/2$ are not considered in this and the next section, so that all cosine-elements in the linearization matrix are different from zero.

5.3. Several theorems on stability

Theorem 5. *A phase locking solution defined by phase differences ϕ_i belonging to $(-\pi/2, \pi/2)$ is asymptotically stable.*

Proof. If all $\phi_j \in (-\pi/2, \pi/2)$, i.e., if all diagonal elements of J are smaller than zero, all Gershgorin discs lie in the closed left half plane, implying that all eigenvalues of J lie in the closed left half plane as well.

The zero-order term of the characteristic polynomial (14) is zero, implying that at least one eigenvalue is zero. The coefficient corresponding to the first-order term is

$$\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} \cos\phi_k. \tag{15}$$

If $\phi_j \in (-\pi/2, \pi/2), \forall j \in \underline{N}$, this coefficient is strictly positive. Exactly one eigenvalue is zero and the remaining eigenvalues are located in the open left half plane. Local asymptotic stability of the corresponding phase locking solution follows. □

Theorem 6. *A phase locking solution with two or more phase differences ϕ_i inside $(\pi/2, 3\pi/2)$ is unstable if $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} \cos\phi_k \neq 0$.*

Proof. If there exist two or more phase differences belonging to $(\pi/2, 3\pi/2)$ then two or more Gershgorin discs lie in the closed right half plane. Because of theorem 4, two or more eigenvalues are located in the closed right half plane. If $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} \cos\phi_k \neq 0$, then exactly one eigenvalue is zero, which implies that one or more eigenvalues have a strictly positive real part. The corresponding phase locking solution is locally unstable. □

Theorem 7. *If the ring consists of identical oscillators, a phase locking solution with two or more phase differences ϕ_i inside $(\pi/2, 3\pi/2)$ is unstable.*

Proof. In the identical oscillator case a phase locking solution is of the form

$$\phi = (\underbrace{\alpha, \ldots, \alpha}_{m}, \underbrace{\pi - \alpha, \ldots, \pi - \alpha}_{N-m}),$$

or a permutation thereof, with

$$m\alpha + (N - m)(\pi - \alpha) = 2\pi k, \qquad k \in T, \quad m \in \underline{N}. \tag{16}$$

If $m \neq N/2$, it is easy to derive that $\sum_{j=1}^{N} 1/\cos\phi_j \neq 0$. Since

$$\left(\prod_{i=1}^{N} \cos\phi_j\right)\left(\sum_{j=1}^{N} \frac{1}{\cos\phi_j}\right) = \sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} \cos\phi_k,$$

theorem 6 is applicable, yielding instability.

If $m = N/2$, then $\sum_{j=1}^{N} 1/\cos\phi_j = 0$. The linearization possesses at least two zero-eigenvalues. Substituting $m = N/2$ in (16) yields $N = 4k, k \in T$. Since $m = N/2$, it is possible to perform a permutation of the phase differences such that the matrix J transforms into $-J$. It has been shown in section 5.2 that the eigenvalues of the linearization are invariant under such a permutation, implying that the set of eigenvalues of J is equal to the set of eigenvalues of $-J$. Hence, if λ is an eigenvalue of J, then so is $-\lambda$. The matrix J has some nonzero-eigenvalues, since it is different from the null matrix. Gershgorin's theorem shows that these nonzero-eigenvalues are not located on the imaginary axis. From this it can be concluded that at least one of the eigenvalues of J has a strictly positive real part. This proves the instability of the corresponding phase locking solution. □

Theorem 8. *If $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} \cos\phi_k < 0$, the phase locking solution with exactly one phase difference ϕ_i belonging to $(\pi/2, 3\pi/2)$ is locally unstable. If $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} \cos\phi_k > 0$, such a solution is stable.*

Table 1. Summary of the stability theorems.

Phase locking solution with	Stability property
No phase differences $\in [\pi/2, 3\pi/2]$	Stable
One phase difference $\in (\pi/2, 3\pi/2)$ and $\sum_{j=1}^{N}\prod_{k=1,k\neq j}^{N}\cos\phi_k > 0$	Stable
One phase difference $\in (\pi/2, 3\pi/2)$ and $\sum_{j=1}^{N}\prod_{k=1,k\neq j}^{N}\cos\phi_k < 0$	Unstable
Two or more phase differences $\in (\pi/2, 3\pi/2)$ and $\sum_{j=1}^{N}\prod_{k=1,k\neq j}^{N}\cos\phi_k \neq 0$	Unstable
One or more phase differences $\in (\pi/2, 3\pi/2)$ and identical oscillators	Unstable

Proof. If a phase locking solution possesses exactly one phase difference ϕ_i belonging to $(\pi/2, 3\pi/2)$, exactly one Gershgorin disc lies in the closed half plane. At most one eigenvalue is positive. It can be shown that the coefficient (15) belonging to the first-order term of the characteristic polynomial is equal to

$$(-1)^{N-1}\prod_{i=1}^{N-1}\lambda_i,$$

where λ_i are the eigenvalues of J from which one zero-eigenvalue is excluded. Now,

$$\mathrm{Re}(\lambda_i) < 0, \qquad \forall i \in \underline{N-1} \quad\Rightarrow\quad (-1)^{N-1}\prod_{i=1}^{N-1}\lambda_i > 0,$$

$$\mathrm{Re}(\lambda_i) \neq 0 \qquad \forall i \in \underline{N-1} \text{ and } \exists!\lambda_i : \mathrm{Re}\,(\lambda_i) > 0 \quad\Rightarrow\quad (-1)^{N-1}\prod_{i=1}^{N-1}\lambda_i < 0.$$

Hence, the linearization is unstable if

$$\sum_{j=1}^{N}\prod_{k=1,k\neq j}^{N}\cos\phi_k < 0,$$

and stable if

$$\sum_{j=1}^{N}\prod_{k=1,k\neq j}^{N}\cos\phi_k > 0.$$

□

Theorem 9. *If all oscillators are identical then each phase locking solution with one phase difference ϕ_i belonging to $(\pi/2, 3\pi/2)$ is locally unstable.*

Proof. From theorem 1 it follows that phase locking solutions with the above property are determined by a permutation of the vector $\phi = (\alpha, \ldots, \alpha, \pi - \alpha)$, with $\alpha \in (-\pi/2, \pi/2)$ satisfying (3). The coefficient of the first-order term of the characteristic polynomial (15) then is $(2 - N)\cos^{N-1}\alpha$. For $N > 2$ this is strictly negative since $\cos\alpha > 0$. According to theorem 8 the corresponding phase locking solution is unstable. □

The above stability theorems are summarized in table 1.

Some remarks:

(i) It can be proven that for each configuration of natural frequencies there always exist phase locking solutions with no phase differences belonging to $[\pi/2, 3\pi/2]$.

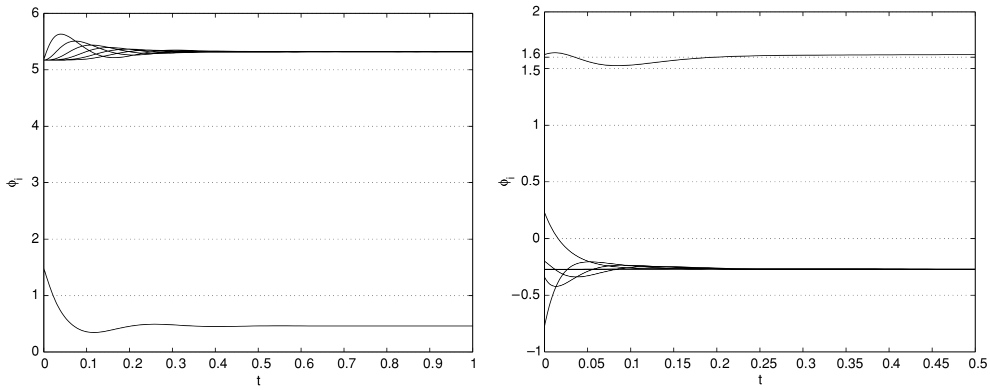

Figure 6. The evolution in time of the seven phase differences $\phi_i \triangleq \theta_i - \theta_{i-1}, i = 1, \ldots 7$, of the ring consisting of seven oscillators with the following properties: the coupling strength is $K = 39.5$ and the vector of natural frequencies is $(0, 0, 0, 0, 0, 0, -50)$. (*a*) No phase differences $\in (\pi/2, 3\pi/2)$. (*b*) One phase difference $\in (\pi/2, 3\pi/2)$.

(ii) Simulations show that for some, but not all, configurations of natural frequencies, there exist *stable* phase locking solutions belonging to the class treated in theorem 8. This is surprising, because it implies a qualitative difference between the identical and the nonidentical oscillator case. Figure 6 shows a simulation of all stable phase locking solutions of a unidirectionally coupled ring consisting of seven oscillators. The natural frequencies of the oscillators are all zero, except $\omega_7 = -50$, and the coupling strength is 39.5. The solution on the left-hand side belongs to the class investigated in theorem 5, whereas the (stable) solution on the right-hand side is considered in theorem 8. To which of the two solutions the state of the system converges depends on the initial conditions.

(iii) The above stability analysis is applicable in the more general case of non-uniform coupling:

$$\dot{\theta}_i = \omega_i + K_i \sin(\theta_{i+1} - \theta_i), \tag{17}$$

with $K_i \geqslant 0$. The consistency condition (11) changes into

$$\sum_{j=1}^{N} g_j \left(\frac{\Omega - \omega_j}{K_j} \right) = 2\pi k, \tag{18}$$

but the stability criteria remain unaltered.

(iv) The approach to the stability analysis by investigation of the linearization prohibits the study of a number of phase locking solutions. First, if a phase locking solution is such that its phase differences satisfy $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} \cos\phi_k = 0$, the linearization possesses multiple zero-eigenvalues. In most cases the presence of positive eigenvalues cannot be concluded. Local stability cannot be analysed in this way.

Second, we assumed that the phase locking solutions do not possess phase differences equal to $\pi/2$ or $3\pi/2$. This assumption excludes all phase locking solutions of which the rank of the linearization is smaller than or equal to $N - 2$. These solutions have again multiple eigenvalues. For instance, the linearization of the travelling wave solution $\phi = (\pi/2, \pi/2, \pi/2, \pi/2)$ of the identical four-oscillator ring is a matrix with exclusively zero-entries, yielding no information whatsoever about stability. This results from the specific form of the coupling, namely, sine coupling. Instead of pursuing the stability of

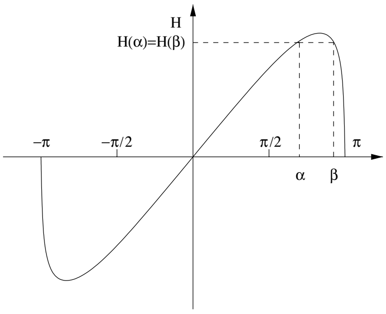

Figure 7. Generalized coupling function.

Table 2. Stability results for generalized coupling.

Phase locking solution with	Stability property
All phase differences $\in (-\gamma, \gamma)$	Stable
One phase difference $\in (\gamma, 2\pi - \gamma)$ and $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} H'(\phi_k) > 0$,	Stable
One phase difference $\in (\gamma, 2\pi - \gamma)$ and $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} H'(\phi_k) < 0$,	Unstable
Two or more phase differences $\in (\gamma, 2\pi - \gamma)$ and $\sum_{j=1}^{N} \prod_{k=1,k\neq j}^{N} H'(\phi_k) \neq 0$	Unstable

those remaining solutions, we generalize the coupling function in the next section; as a bonus we are able to deal with solutions of the type $\phi = (\pi/2, \pi/2, \pi/2, \pi/2)$.

6. Generalized coupling

For the unidirectionally coupled ring it is possible to generalize the study from sine coupling to a broader class of coupling functions. The properties of the coupling functions H we consider are the following. The function H is continuous, 2π-periodic and odd, and has exactly one maximum belonging to $(0, \pi)$. An example is shown in figure 7. Phase locking solutions are obtained in a similar way as described in section 3. Consider a network with identical oscillators:

$$\dot{\theta}_i = K H(\theta_{i+1} - \theta_i), \qquad i \in \underline{N}. \tag{19}$$

Theorem 10. *Consider all triples* (α, β, m), $m \in \underline{N}$, $\alpha, \beta \in S^1$, $(\alpha \neq \beta)$, *satisfying*

$$m\alpha + (N - m)\beta = 2\pi k, \qquad k \in T,$$
$$H(\alpha) = H(\beta).$$

Assign to each such triple (α, β, m) *the vector*

$$(\underbrace{\alpha, \ldots, \alpha}_{m}, \underbrace{\beta, \ldots, \beta}_{N-m}).$$

Every vector $\phi = (\phi_1, \ldots, \phi_N)$ *that is a permutation of such a vector (corresponds to a phase locking solution of (19).*

For $m = N$ the travelling wave solutions are retrieved, identical to the travelling wave solutions of (2). The stability properties, however, may differ from the case with sine coupling, as follows from the theorems below. With $K > 0$ and $\gamma \in (0, \pi) : H'(\gamma) = 0$, the results in table 2 can be proven.

7. Application to antenna arrays

The system equations proposed in Kuramoto (1984) are closely related to those arising in antenna array theory (York 1993, Liao and York 1993, Navarro and Chang 1996, Pogorzelski *et al* 1999, Dussopt and Laheurte 1999). In an antenna array, each antenna i is driven by a Van der Pol oscillator, described by its complex output voltage $V_i(t) = A_i(t)\,\mathrm{e}^{\mathrm{i}\theta_i(t)}$. Under the assumption of weak coupling, the amplitude of each oscillator A_i is considered to be constant and the dynamics of the phases θ_i becomes (York 1993):

$$\dot{\theta}_i = \omega_i - \frac{\omega_i}{2Q}\sum_{j=1}^{N}\epsilon_{ij}\frac{\alpha_j}{\alpha_i}\sin(\Phi_{ij} + \theta_i - \theta_j), \qquad \forall i \in \underline{N},$$

with Q the quality factor of the coupled network, α_i the (constant) free-running amplitude of the ith oscillator, ω_i the free-running frequency of the ith oscillator and $\epsilon_{ij}\,\mathrm{e}^{\Phi_{ij}}$ the complex coupling between oscillators i and j. The coupling amplitude ϵ_{ii} is assumed zero, without loss of generality. Assume that each antenna is coupled with the others in an identical way, then

$$\epsilon_{ij}\,\mathrm{e}^{\Phi_{ij}} = \epsilon\,\mathrm{e}^{\Phi}, \qquad \forall i, j \in \underline{N}.$$

A simplification often made (York 1993, Pogorzelski *et al* 1999) is replacing the factor $(\omega_i\epsilon)/(2Q)$ in each equation by a common factor $\Delta\omega$, called the locking range of the oscillator. This simplification is only valid if the frequency differences are sufficiently small.

The system equations can be further simplified by assuming identical free-running amplitudes α_i of the oscillators (Liao and York 1993). This yields a dynamics

$$\dot{\theta}_i = \omega_i - \Delta\omega\sum_{j=1}^{N}\sin(\Phi + \theta_i - \theta_j), \qquad \forall i \in \underline{N}.$$

When coupled in a unidirectional ring these equations become

$$\dot{\theta}_i = \omega_i + \Delta\omega\sin(\Phi + \theta_{i+1} - \theta_i), \qquad i \in \underline{N}.$$

All phase locking solutions can be obtained using the same technique as in section 4. The stability theorems proven in section 5.2 can be modified.

Theorem 11. *If a phase locking solution has only phase differences ϕ_i belonging to $(-\pi/2 + \Phi, \pi/2 + \Phi)$, then it is asymptotically stable.*

In Dussopt and Laheurte (1999) an antenna array of four *identical* unidirectionally coupled oscillators is investigated. The coupling is made unidirectional by inserting unilateral amplifiers between the oscillators, as explained in Lin *et al* (1994). Each antenna is linearly polarized. The array is constructed in such a way that if the phase locking solution $\phi = (\pi/2, \pi/2, \pi/2, \pi/2)$ can be maintained, the antenna array emits a circularly polarized wave. Using Maple all stable phase locking solutions were determined, together with a stability condition on the coupling phase Φ.

In the present paper *all* phase locking solutions and their stability properties are determined analytically. The stability condition on ϕ for the travelling wave to be stable is analytically proven. The present paper also shows that the setting of Dussopt and Laheurte (1999) can be generalized to nonidentical oscillators. If the oscillators are not identical and if the coupling is sufficiently strong there exist phase locking solutions which are locally stable. Because of (9), mutually different frequencies ω_i yield mutually different phase differences ϕ_i. Hence the travelling wave solutions (7) do not belong to the set of solutions and the polarization of the wave created by the oscillator will not be perfectly circular. However simulations show that for K sufficiently large, there exist phase locking solutions with values of ϕ_i very close to those of the travelling wave. Once the values of ϕ_i are known, theorem 11 yields a condition on Φ for the phase locking solution to be stable.

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